A SIMPLE PROOF OF THE ATIYAH-SEGAL COMPLETION THEOREM

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Abstract. My research focuses on the Atiyah-Segal completion theorem. This is a theorem in equivariant K theory in mathematics. It relates completion of K group with K group of completion space.

To be explicit, let me give some standard notations in topology and algebra: suppose G is a finite group for simplicity. The spaces we study are CW complexes. Here a CW complex means a space that can be written as a sequence of attaching disks. Furthermore, a G-vector bundle means attaching disk orbits $(G/H) \times D^k$. K theory studies the vector bundles over a space, which is a locally trivial projection with each fiber a vector space. The direct sum of two vector bundles gives an addition on the set of vector bundles and it forms a monoid. The K group of a space is defined by making this direct sum a group by what is called Grothendieck group. For any group G, there exists a universal bundle $EG \to BG$ such that every bundle can be obtained by pull-backs. Then the completion space is defined as $X \times EG$.

This work follows from a 1988 paper. My research aims to add details to the proof in the paper in a simple and clear mannar.

1. Introduction

Paper "A Generalization of the Atiyah-Segal Completion Theorem" gives a shortest route from Bott periodicity(showed and explained in the proof of Lemma 2.11 in this report) to a generalized version of the Atiyah-Segal completion theorem[1]. The Atiyah-Segal completion theorem generates to completion at a family of subgroups. Basic progroup language is used in the proof of that paper. In this paper there will be a more accessible proof explaining firstly the simple progroup language in section 3 and then the topological part of the proof in section 4.

In order to make the proof more explicit, the focus on this research will be the complex number case since the proof for the real number case is identical to the complex number case. Also, we will focus only on the original form of the Atiyah-Segal Completion Theorem showed below, and ignore the generalization to a family of subgroups that is mentioned in the article by Adams et al. (1988)[1]. The notations of the theorem can be found in the next section.

Theorem 1.1. *(The Atiyah-Segal Completion Theorem) Let X be a G-space. Then the projection $EG \times X \to X$ induces an isomorphism of progroup $K^*_G(X)_{\widehat{I}} \to K^*_G(\mathrm{EG} \times X)$.*

2. Method Part

2.1. **Equivariant K-theory.** In this paper, the definition of equivariant K theory as suggested in Segal's publication(1968) [5] will be used.

Let G be a compact Lie group and we understand a G-space as a G-CW-complex. Define a G-vector bundle to be a G-map $p : E \to X$ which is a complex vector bundle. For any $g \in G$ and $x \in X$, $g : E_x \to E_{gx}$ is a homomorphism of vector spaces. Let $K_G(X)$ be the Grothendieck group of the monoid of G-vector bundles over some G-space X.

Call two G-vector bundles E, E' *stably equivalent* if there exist trivial G-vector bundles M and M' such that $E \oplus M \cong E' \oplus M'$. The stable equivalence classes form an abelian group $\tilde{K}_G(X)$. For example, $R(G)$, which is the representation ring of G, is also equal to $K_G(\text{pt})$, and $\tilde{K}_G(pt) = 0$ since we only have trivial G-vector bundles over a point.

Define $\tilde{K}_G^{-k}(X) = \tilde{K}_G(S^k X)$, $\tilde{K}_G^{-k}(X, A) = \tilde{K}_G(S^k(X \cup_A CA))$, $K_G^{-k}(X) = \tilde{K}_G^{-k}(X_+)$ and $\tilde{K}_G^{-k}(X, A) = \tilde{K}_G^{-k}(X_+, A_+)$ where X_+ is the one-point compactification of X and $S^k X$ is the k-th suspension which is equal to the smash product $S^k \wedge X$. For example, $K_G^n(pt) = 0$ if n is odd and $K_G^n(pt) = R(G)$ if n is even by equivariant Bott periodicity explained in section 4. Furthermore, Segal(1968) shows that $K_G^*(X)$ is a generalized cohomology theory and $\tilde{K}_G^*(X)$ is a reduced generalized cohomology theory.

Let EG be the universal space of G, $E^n G$ be the n-skeleton of EG. Then EG is the geometric realization of the simplicial space $E_n(G) = G^{n+1}$ with certain faces and degeneracies. So EG is a G-CW complex. Explicit construction and properties can be find in 16.5 of May's book(1999) [3]. Define I_G to be the augmentation ideal of $R(G)$, that is, the kernel of the restriction $R(G) \to R(\{e\})$. We will simply denote I_G as I if there is no other group causing ambiguity.

In order to prove the Atiyah-Segal completion theorem, the theorem is given in the paper by Adams et al.[1](see Theorem 2.1 below) will be proved in section 2.3.

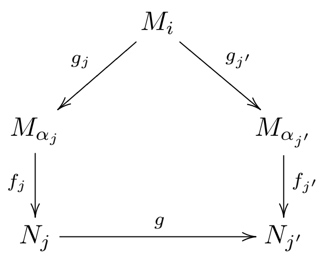
Theorem 2.1. *If X and Y are G-spaces, and a G-map $f : X \to Y$ is a homotopy equivalence, then f induces an isomorphism $K_G^*(Y)_{\hat{I}} \to K_G^*(X)_{\hat{I}}$.*

2.2. **Progroups.**

Definition 2.2. A *progroup* is an inverse system of Abelian groups, indexed on a filtered directed poset.

Definition 2.3. If $\{M_\alpha\}$, $\{N_\beta\}$ are two progroups, then define the homomorphism set $\text{Hom}(\{M_\alpha\}, \{N_\beta\}) = \varprojlim_\beta \varinjlim_\alpha \text{Hom}(M_\alpha, N_\beta)$.

To be explicit, an arrow in $\text{Hom}(\{M_\alpha\}, \{N_\beta\})$ can be represented by a set $\{f_j : M_{\alpha_j} \to N_j\}$ of group homomorphisms, one for each j, such that for each arrow $g : N_j \to N_{j'}$ of $\{N_\beta\}$, there is some i, an arrow $g_j : M_i \to M_{\alpha_j}$ and an arrow $g_{j'} : M_i \to M_{\alpha_{j'}}$ such that $g \circ f_j \circ g_j = f_{j'} \circ g_{j'}$.



Note that f_j is a representative of an equivalence class in $\varinjlim_\alpha \mathrm{Hom}(M_\alpha, N_j)$. Each such f_j is called a *representative of* f. On the other hand, two sets $\{f_j : M_{\alpha_j} \to N_j\}$ and $\{f'_j : M_{\alpha'_j} \to N_j\}$ are representatives of the same arrow if for every j, there exists some i, an arrow $g_j : M_i \to M_{\alpha_j}$ and an arrow $g'_j : M_i \to M'_{\alpha_j}$ such that $f_j \circ g_j = f'_j \circ g'_j$.

Then it gives a unique composition of two morphisms, and the collection of identity morphisms in $\mathrm{Hom}(M_\alpha, M_\alpha)$ gives the identity morphism of $\{M_\alpha\}$. As a result, progroups form a category.

Generally, a progroup contains more information than the limit of the groups. Also, $\{0\}$ gives a zero progroup.

Proposition 2.4. *A progroup $\{M_i\}$ is isomorphic to zero if and only if for any i, there exists a zero homomorphism $f_i : M_j \to M_i$. Such progroup is called pro-zero.*

Proof. A progroup $\{M_i\}$ isomorphic to zero means that id is equal to 0. So by the explanation above, it is equal to the right hand side. □

The appendix of an article by Artin and Mazur(1969) [7] shows that progroups form an Abelian category. An explicit description of exact sequences of progroups can be found below:

Proposition 2.5. *Given a sequence of two morphisms of progroups $\{L_i\} \xrightarrow{f} \{M_j\} \xrightarrow{g} \{N_k\}$ such that the composition is zero. The sequence is pro-exact if for every representative $f_j : L_i \to M_j$ of f, there is some $h_j^m : M_m \to M_j$ and some representative $g_k : M_m \to N_k$ of g such that $h_j^m(\ker(g_k)) \subset \mathrm{im}(f_j)$.*

To give the statement of the Atiyah-Segal completion theorem, we need the notion of the *formal completion.* Given a ring R, an ideal I and a progroup $\{M_i\}$ with index A which consists of R-modules.

Definition 2.6. Define the *I-adic completion* of $\{M_i\}$ to be progroup $\{M_i/I^k M_i \mid i \in A, k \in \mathbb{Z}_+\}$ with index $A \times \mathbb{Z}_+$, and denote it as $\{M_i\}_{\widehat{I}}$.

Then it is an exact functor as follows:

Lemma 2.7. *Suppose R is a Noetherian ring, I is an ideal of R and $f : A \to B$ is a morphism between finitely generated R-modules. Then there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}_+$, $\ker(A \to B/I^{n+c}B) \subset \ker(f) + I^n A$.*

Proof. We can decompose $A \to B/I^{n+c}B$ into $A \to \mathrm{im}(f) \to B/I^{n+c}B$ and let C be $\mathrm{im}(f)$. Then $0 \to C \to B$ is exact. By Artin-Rees lemma, we can choose $c > 0$ such that for any $n > 0$, $\ker(C \to B/I^{n+c}B) = C \cap I^{n+c}B \subset I^n C$.

Since tensoring $R/I^n R$ is right exact and $C = \mathrm{coker}(\ker(f) \to A)$, $\ker(f)/I^n\ker(f) \to A/I^n A \to C/I^n C \to 0$ is exact. So $\ker(A \to C/I^n C) \subset \ker(f) + I^n A$. Combining the two results we come to the conclusion that $\ker(A \to B/I^{n+c}B) \subset \ker(f) + I^n A$. □

Theorem 2.8. *If R is a Noetherian ring, the I-adic completion is an exact functor in the subcategory of progroup consisting of finitely generated R-modules.*

Proof. For any morphism $f : M \to N$ between R-modules, $f(I^k M) \subset I^k N$. Let $\tilde{f}^k$ be the corresponding morphism $M/I^k M \to N/I^k N$. It commutes with composition. So formal completion is a functor.

Then it suffices to prove that for pro-exact sequence $\{A_i\} \xrightarrow{f} \{B_j\} \xrightarrow{g} \{C_k\}$, the sequence $\{A_i\}_{\widehat{I}} \to \{B_j\}_{\widehat{I}} \to \{C_k\}_{\widehat{I}}$ is pro-exact.

For any representative $f_j : A_i \to B_j$ and any $m \in \mathbb{Z}_+$, suppose there is $r_j^{j'}$: $B_{j'} \to B_j$ and $g_k : B_{j'} \to C_k$ such that $r_j^{j'}(\ker(g_k)) \subset \mathrm{im} f_j$. By the lemma above, we have some $c > 0$. Suppose $\tilde{g}_k$ is the map $B_{j'}/I^{m+c}B_{j'} \to C_k/I^{m+c}C_k$ induced from g_k. Then we have that $\ker(\tilde{g}_k) \subset \ker(g_k) + I^m$. Let $\tilde{f}_j$ be $A_i/I^mA_i \to B_j/I^mB_j$ induced from f_j and $\tilde{r}$ be $B_{j'}/I^{m+c} \to B_j/I^m$ induced from $r_j^{j'}$. Then $\mathrm{im}(\tilde{f}_j) = \mathrm{im}(f_j) + I^m$. So by Lemma 2.7, $\tilde{r}(\ker(\tilde{g}_k)) \subset \tilde{r}(\ker(g_k) + I^mB_{j'}) = r(\ker(g_k)) + I^mB_j \subset \mathrm{im}(\tilde{f}_j)$. So the sequence $\{A_i\}_{\widehat{I}} \to \{B_j\}_{\widehat{I}} \to \{C_k\}_{\widehat{I}}$ is pro-exact. □

If X is a finite G-CW-complex, then $X \to \{\mathrm{pt}\}$ gives $R(G) = K_G^0(pt) \to K_G^0(X)$. So $K_G^*(X)$ has the structure of a $R(G)$-module.

Definition 2.9. For any G-CW-complex X(not necessarily finite), $K_G^n(X)$ is a progroup $K_G^n(X_\alpha)$ where X_α runs over the finite G-subcomplexes of X.

Then by the theorem above, $K_G^*(X)_{\widehat{I}}$ is a generalized cohomology theory.

2.3. **Proof of Theorem 2.1.** Let M be a term in the cofiber sequence $X \to Y \to M$. Since X is homotopy equivalent to Y, M is contractible. So by the cofiber exact sequence it suffices to prove the following theorem.

Theorem 2.10. *If X is contractible, then $\tilde{K}_G^*(X)_{\widehat{I}}$ is pro-zero.*

Suppose U is the collection of G representations V such that $V^G = \{0\}$ and $V^H \neq \{0\}$ for some proper subgroup $H < G$. For any proper subgroup $H < G$, notice that the induced representation $\mathrm{Ind}_H^G \mathbb{1}_H$ is nontrivial, so we can choose a nontrivial irreducible sub-G-representation $V \subset \mathrm{Ind}_H^G \mathbb{1}_H$. Then by Frobenius reciprocity, $\mathrm{Hom}_H(\mathbb{1}_H, \mathrm{Res}_H^G V) = \mathrm{Hom}_G(\mathrm{Ind}_H^G \mathbb{1}_H, V)$. Thus $V^G = \{0\}$ and $V^H = V \neq \{0\}$. So we have $V^{\oplus k} \in U$ for any positive integer k. Let I be a finite set of representations in U, and let Y^I be the one point compactification of the direct sum of elements in I. Then $(Y^I)^G = S^0$. The inclusion $I \subset J$ gives an inclusion $Y^I \to Y^J$. Define Y to be the colimit of all Y^I's.

Lemma 2.11. $\{\tilde{K}_H^*(Y)\}_{\widehat{I_H}}$ *is pro-zero for all $H < G$.*

Proof. If H is a proper subgroup of G, then for Y^I, choose V' such that H acts trivially on V'. Let J be $I + \{V'\}$, V be the direct sum of elements in I, and W be $V \oplus V'$. Then $Y^I = S^V$ and $Y^J = S^W$. So $((v, v'), t) \mapsto (v, v'/t)$ for $0 < t \leq 1$, $((v, v'), 0) \mapsto \infty$, and $(\infty, t) \mapsto \infty$ give a homotopy between the null map to infinity and the inclusion map from S^V to S^W. So such inclusion map is null-homotopic. Since I is arbitrary, Y is H-contractible. Then the lemma is obvious.

If $H = G$, it suffices to show that for any Y^I and m, there exists $Y^I \to Y^J$ such that the map $\tilde{K}_H^*(Y^J)/I^m\tilde{K}_H^*(Y^J) \to \tilde{K}_H^*(Y^I)/I^m\tilde{K}_H$ is 0.

The Bott periodicity for equivariant K theory is described at Proposition 3.2 of Segal's publication[5] and Theorem 4.3 of the paper by Atiyah[8]. It says the following: for any complex G-module V, there is an element $\lambda_V \in K_G^0(V) = \tilde{K}_G^0(S^V)$, $\lambda_V = \sum_{i=0}^{\infty}(-1)^i\Lambda^iV$ where Λ^i is the i-th wedge power. Then multiply by λ_V induces an isomorphism $K_G^*(X) \to K_G^*(V \times X)$. If $W = V \oplus V'$, then we have $\lambda_W = \lambda_{V'}\lambda_V$.

By Bott periodicity, taking $X = \mathrm{pt}$, then $\tilde{K}_G^*(S^V)$ is the free $\tilde{K}_G^*(S^0)$-module generated by the Bott class $\lambda_V \in \tilde{K}_G^0(S^V)$. Suppose $W = V \oplus V'$. The inclusion $i : S^V \to S^W$ is equal to $\mathrm{id}_V \wedge i'$ where i' is the inclusion $S^0 \to S^{V'}$. Then $i^*(\lambda_W) = i'^*(\lambda_{V'})\lambda_V$. Notice that there is a commutative diagram

$$\begin{array}{ccc} \tilde{K}_G^n(S^V) & \xrightarrow{i'^*_G} & \tilde{K}_G^n(S^0) \\ \downarrow r & & \downarrow r \\ \tilde{K}^n(S^V) & \xrightarrow{i'^*_e} & \tilde{K}^n(S^0) \end{array}$$

where each column is the restriction. Since i' is null homotopic (not equivariantly), $r(i'^*_G(\lambda_{V'})) = i'^*_e(r(\lambda_{V'})) = 0$. So $i'^*(\lambda_{V'}) \in I$. Choose $Y^J = Y^I \oplus V \oplus \cdots \oplus V$(the number of V's is m) then the proof is done. □

There is a cofiber sequence $S^0 \to Y^I \to Y^I/S^0$ and take smash product with some X to get a cofiber sequence $X \to X \wedge Y^I \to X \wedge (Y^I/S^0)$.

Theorem 2.12. *For any finite G-CW-complex X, the progroup $\{\tilde{K}_G^*(Y \wedge X)_{\hat{I}} \mid k \in \mathbb{N}\}$ is zero.*

Proof. Use induction on dimension d of X. The case when $d = 0$ follows from Lemma 2.11. It suffices to show that the G-space X' obtained by attaching an d cell to X still satisfies the property. There is a pro-exact sequence

$$\cdots \to \tilde{K}_G^*((G/H)_+ \wedge S^d \wedge Y)_{\hat{I}} \to \tilde{K}_G^*(X')_{\hat{I}} \to \tilde{K}_G^*(X)_{\hat{I}} \to \cdots$$

. By the suspension axiom, it suffices to show that $\tilde{K}_G^*((G/H)_+ \wedge Y)_{\hat{I}}$ is pro-zero. Example (iii) of section 2 in Segal's publication(1968)[5] shows that $\tilde{K}_G^*((G/H)_+ \wedge Y) = \tilde{K}_H^*(Y)$. Corollary 3.9 of Segal's paper(1968)[4] shows that I_G (acts through restriction) and I_H topology are the same. Then it follows from pro-exactness and lemma 2.11. □

Since M is contractible, we consider the homotopy $H : M \times [0,1] \to E_G$ such that $H(M \times \{0\}) = \mathrm{id}_M$ and $H(M \times \{1\}) = \mathrm{pt}$. Since every G-subcomplex M^n has a open neighborhood that contracts to itself and it is compact, $H(M^n \times [0,1]) \subset M^m$ for some m. So $M^n \hookrightarrow M^m$ is null homotopic. As a result, the progroup $\tilde{K}^*(M)$ is zero.

We also need a remark to prove $\tilde{K}_G^*(M)_{\hat{I}}$ is pro-zero.

Remark 2.13. Consider the lexicographical ordering given by dimension and the number of connected components of subgroups of the compact Lie group G. We have that any descending chain of G is finally constant. So we can use transfinite induction on the poset of inclusion of subgroups of G.

Theorem 2.14. *For any G-space X such that $\tilde{K}^*(X)$ is pro-zero, we have $\tilde{K}_G^*(X)_{\hat{I}}$ is pro-zero.*

Proof. We use induction on subgroup $H < G$.

The case $H = e$ is just the hypothesis.

Suppose H is not e. Without loss of generosity let $H = G$. Cofiber sequence $X \to X \wedge Y \to X \wedge (Y/S^0)$ gives a long pro-exact sequence

$$\cdots \to \tilde{K}_G^k(X \wedge Y)_{\hat{I}} \to \tilde{K}_G^k(X)_{\hat{I}} \to \tilde{K}_G^{k+1}(X \wedge (Y/S^0))_{\hat{I}} \to \cdots$$

By Theorem 2.12, $\tilde{K}_G^k(X \wedge Y)_{\hat{I}}$ is pro-zero. So it suffices to show that $\tilde{K}_G^k(X \wedge (Y/S^0))$ is pro-zero.

In fact we can prove that $(\tilde{K}_G^k(X \wedge Z))_{\hat{I}}$ is pro-zero for every k and finite G-space Z such that $Z^G = pt$. We use the same induction as Theorem 2.12. We induct on the dimension d of Z. Notice that $Z^0 = pt$, the $d = 0$ case is trivial. Consider attaching a d-cell $(G/H)_+ \wedge S^d$ to Z and getting Z'. Since $Z^G = pt$, $H \neq G$. There is a pro-exact sequence

$$\cdots \to \tilde{K}_G^k((G/H)_+ \wedge S^d \wedge X)_{\hat{I}} \to \tilde{K}_G^k(Z' \wedge X)_{\hat{I}} \to \tilde{K}_G^k(Z \wedge X)_{\hat{I}} \to \cdots$$

. By the suspension axiom, it suffices to show that $\tilde{K}_G^k((G/H)_+ \wedge X)_{\hat{I}}$ is pro-zero for each k. Since $\tilde{K}_G^*((G/H)_+ \wedge X)_{\widehat{I_G}} = \tilde{K}_H^*(X)_{\widehat{I_H}}$ and H is a proper subgroup of G, it follows from the induction hypothesis. □

Combining Theorem 2.14 and the discussion above we get Theorem 2.10.

3. Results

To deduce Theorem 2.1 from Theorem 2.2, it suffices to show that $K_G^*(EG \times X)$ is I-adically complete.

Lemma 3.1. *If G acts freely on some G-space X, then $K_G^*(X)$ is discrete in the I-adic topology, so it is complete.*

This lemma can be found at Proposition 4.3 in the article by Atiyah and Segal(1969)[6].

From the construction of E^nG we know that G acts freely on EG. So G acts freely on $EG \times X$. Then Theorem 2.1 holds.

If we take $X = \text{pt}$, then we have that $K(BG) = K_G(EG) \cong R(G)_{\hat{I}}$.

If X is compact, then $K_G(EG \times X) \cong K_G(X)_{\hat{I}}$. So $K_G(EG \times X)$ satisfies the Mittag-Leffler condition. Then we can identify $K_G^*(EG \times X)$ with $\lim_n K_G^*(E^nG \times X)$.

4. Discussion

By using method in section 2, we can prove the real number case of the completion theorem using equivariant Bott periodicity of the real number representation which is more useful and interesting than the complex number case.

Also, if we include a little knowledge of Lie group, such as classifying space of a family of subgroups (which requires Elmendorf's theorem), there is a case stated in the paper by Adams et al.(1988) [1]. This paper also uses the same method to prove a localization theorem. And furthermore, we can generalize the Atiyah-Segal completion theorem to twisted K-theory.

Acknowledgements

It's my pleasure to thank my mentor at Research Experience for Undergraduate(REU) in the University of Chicago, Oliver Wang, for being so inspiring and patient when introducing me the topological K theory, answering my questions and helping me with my paper. I would also thank Professor J.Peter May for giving me the subject of this paper and taking a lot of time on carefully pointing out the mistakes in my paper, both mathematical and grammatical. Finally, I would also thank Peking University for giving me the chance to attend REU. Furthermore, I would thank Professor May for organizing such an amazing online REU program during this difficult pandemic period.

References

[1] J.F.Adams, J.-P.Haeberly, S.Jackowski and J.P.May. A Generalization of the Atiyah-Segal Completion Theorem. Topology Vol. 27, No. 1, pp. 1-6, 1988

[2] J.F.Adams, J.-P.Haeberly, S.Jackowski and J.P.May. A Generalization of the Segal's Conjecture. Topology Vol. 27, No. 1, pp. 7-21, 1988

[3] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.

[4] G. B. Segal. The Representation Ring of a Compact Lie Group. Inst. Haures Etudes Sci. Publ. Math. 34 (1968), 113-128.

[5] G. B. Segal. Equivariant K-Theory. Publications mathématiques de l'I.H.É.S., tome 34 (1968), p. 129-151

[6] M. F. Atiyah, G. B. Segal. Equivariant K-Theory and Completion. J. Differential Geometry. 3(1969), p. 1-18

[7] M. Artin, B. Mazur. Etale Homotopy. Springer. 1969.

[8] M. F. Atiyah. Bott Periodicity and the Index of Elliptic Operators. Quart. J. Math. Oxford Ser. 19(1983), p. 113-140.