The algebra of magic squares

Luciana Marconi
Schweizer Jugend forscht, Schweiz

Abstract: For over 4000 years, magic squares have fascinated humans. Yet few people know what they are today. The magic squares have played a significant role in many ancient cultures, where they have been used in astrology and as lucky charms for births to mention a few. In ancient China, they were associeted with the Ying-Yang philosophy and people tried to explain the world with magic squares. However, the question still remains, what makes magic squares so special and why did I do research on them?

Magic squares are square arrangements of numbers with the special characteristic that the sum of the numbers in each row, column and diagonal all equals to the same value. Exactly this phenomenon seemed to be magical for many people, as the name suggests. Not only in a philosophical, but also in a mathematical sense, these objects are highly interesting. In this work, I have used mathematics to find out how and when magic squares are formed. My results are nine simple conditions that must be met to create a magic square. Furthermore, I was able to verify rules that are several thousand years old.

1 Introduction

An nth-order magic square is a square with n rows and n columns, in which the numbers from 1 to n^2 are filled, so that the sum of all the numbers in each row, column and the two diagonals is equal (figure 1).

rows \ columns	1	2	3	4	5	↗ 65
1	8	22	11	5	19	→ 65
2	1	20	9	23	12	→ 65
3	24	13	2	16	10	→ 65
4	17	6	25	14	3	→ 65
5	15	4	18	7	21	→ 65
	↓ 65	↓ 65	↓ 65	↓ 65	↓ 65	↘ 65

figure 1. Magic square of 5th order. Every row, column and the two diagonals have equal sums. For example: row 1 in blue: 8+22+11+5+19 = 65.

The oldest known magic squares come from ancient China. It is said that they were first discovered by the Emperor Yu (2205-2198 BC) during a boat trip where he met the divine turtle Hi, on whose back a numbered drawing depicting the first magical square appeared to him. Magic squares have not only been of great importance in ancient China [2]. In India, magic squares were placed on temple walls and were attributed positive magical powers over many centuries [3]. Many famous personalities like the scientist Benjamin Franklin (1706-1790) or the mathematician Leonhard Euler (1707-1783) were fascinated by these objects and spent much time creating and analyzing magic sqaures. Also in art, for example in Albrecht Dürer's work "Melancolia" (1514) (figure 2) or in Johann Wolfgang von Goethe's most famous work "Faust" (1808), a magic square can be recognized [4]. However, for many people, it was not enough to know that magic squares existed, they also wanted to find out how to make them.

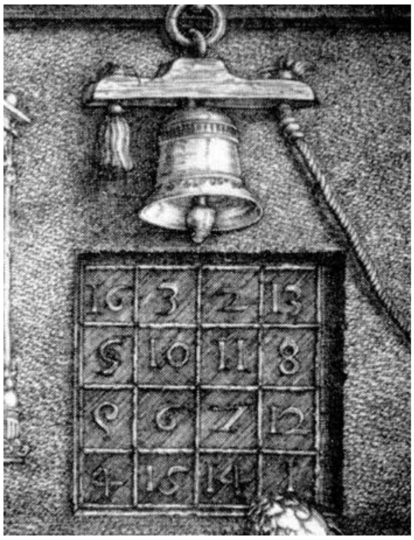

figure 2. The magic square in Dürer's famous art work "Melancolia" [1].

Magic squares are created using different filling methods. This means that you fill the numbers 1 to n^2 into the empty square according to a certain pattern, so that a magic square is created. Some filling methods have been handed down for thousands of years [5].

I will have a closer look at two of the oldest most famous filling methods, namely the "knight's move method" and the "Indian rule". The purpose of this work is to analyse magic squares using mathematical formulas and to find out why and how they are created.

This work was conducted during my final year at high school, where I got the opportunity to delve into my favourite subject and to further expand my knowledge in theoretical mathematics. I decided to reserach on magic squares, because besides the many theoretical proofs, you can also apply your results practically and create magic squares yourself.

2 Method

The first step in this project was to discribe the square, fields and filling methods mathematically. To get the total number of fields in your square, you must multiply the number of fields on two sides, and since we are looking at squares whose sides are equal, you will get n^2 fields, as you can see in figure 3. For example a 3[rd] order square, that means $n = 3$, has $3^2 = 9$ fields. I will now define some variables that I can use later to describe specific filling methods.

The field in which the number 1 is placed, which can usually be placed anywhere you like, is called the **initial field (i_0 | j_0)** (figure 3). From there you move ***a*** fields to the right and ***b*** fields to the top and now you are in the field where the next number is filled (figure 3). Repeat this step until you come across a field that is already filled. If you encounter a field that is already filled, you should move ***s*** fields to the right and ***t*** fields to the top (figure 3). This is done until the whole square is filled in.

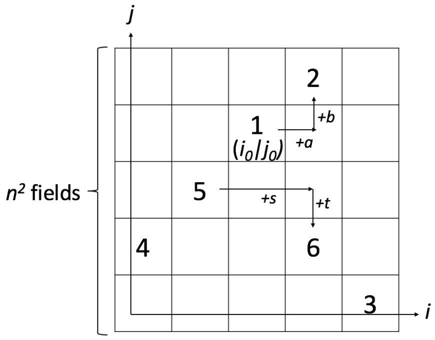

figure 3. Generalisation of filling methods

In a next step, I read up on the theory of algebraic structures and concentrated especially on the residue class ring, which is a perfect tool for my calculations as you will see later. A residue class is the set of integers having the same residue when dividing by n, where n can be any integer. To be able to distinguish between a number and a residue class, I have marked the residue classes underlined in the example below and in formula (1). A residue class ring ($\mathbb{Z}/n\mathbb{Z}$) consists of a set of all residue classes and the operations addition and multiplication.
Example residue class ring:

Residue class ring modulo 5 ($\mathbb{Z}/5\mathbb{Z}$)

$$\begin{cases} \underline{0} = \{0, 5, 10, 15, \ldots\} \\ \underline{1} = \{1, 6, 11, 16, \ldots\} \\ \underline{2} = \{2, 7\ 12, 17, \ldots\} \\ \underline{3} = \{3, 8, 13, 18, \ldots\} \\ \underline{4} = \{4, 9, 14, 19, \ldots\} \end{cases}$$

A special, but for my work important calculation rule of residue rings, is that you can divide by $\underline{a}$ exactly when the greatest common divisor of a and n (gcd(a, n)) equals 1 [6]. The residue class ring is an ideal tool for my calculations, because it only consists of n elements and that is exactly the number of fields in each row, column and diagonal. Therefore, you can never leave the square.

Using these basics, I was able to derive a formula that assigns a number to its appropriate field and vice versa. To get a simpler formula I have to add the variables *y* and *z*. Here *y* is the integer quotient when dividing $X - 1$ (the number to be filled in – 1) by n and z is the residue when dividing $X - 1$ by n.

$$\eta : \quad (y, z) \quad \rightarrow \quad (\underline{i}, \underline{j}) = (\underline{i_0 + (z-y)a + ys}, \underline{j_0 + (z-y)b + yt}) \tag{1}$$

The question now is how to choose all the many variables of the formula (1) so that a magic square is created based on this assignment.

3 Results

By solving different equations, which are based on the formula (1), I have found nine independent conditions that should be taken into account when choosing the variables to construct a magic square, namely:

1) $\gcd(at - bs, n) = 1$
2) $\gcd(b, n) = 1$
3) $\gcd(b - t, n) = 1$
4) $\gcd(a, n) = 1$
5) $\gcd(a - s, n) = 1$
6) $\gcd(a - b, n) = 1$
7) $\gcd(a - s + t - b, n) = 1$ } replaceable by: $\underline{i_0} = \underline{j_0} + \underline{(s-t)} \cdot \underline{2}^{-1}$ in $\mathbb{Z}/g_A\mathbb{Z}$
8) $\gcd(a + b, n) = 1$
9) $\gcd(a - s + b - t, n) = 1$ } replaceable by: $\underline{i_0} = \underline{(t+s)} \cdot \underline{2}^{-1} - \underline{j_0} - \underline{1}$ in $\mathbb{Z}/g_B\mathbb{Z}$

g_A and g_B are defined as:

- $g_A = \gcd(a - b, n) \bullet \gcd(a - s + t - b, n)$
- $g_B = \gcd(a + b, n) \bullet \gcd(a - s + b - t, n)$

From the first five conditions it can be deduced that n must be an odd number. If all conditions 1 to 9 are fulfilled, you can create a magic square, regardless of which field you start in. If one or more of the last four conditions are not met, the alternative conditions can be used to calculate the appropriate initial field, so that a magic square is still formed.

As I mentioned in the introduction, I have analysed two old and very well-known filling methods, "knight's move method" and "Indian rule", based on my results and we will see that they both meet my conditions.

The name "knight's move method" comes from the fact that when filling the numbers, you proceed like the knight from chess. This method works for any square whose n is an odd number and not a multiple of 3.
Example: The variables are chosen as follows: $n = 5$, $a = 2$, $b = 1$, $s = 0$, $t = 2$. In this case all the conditions are fulfilled. How to fill the numbers using the knights filling methods and the resulting magical square can be seen in figure 4 A and B respectively.

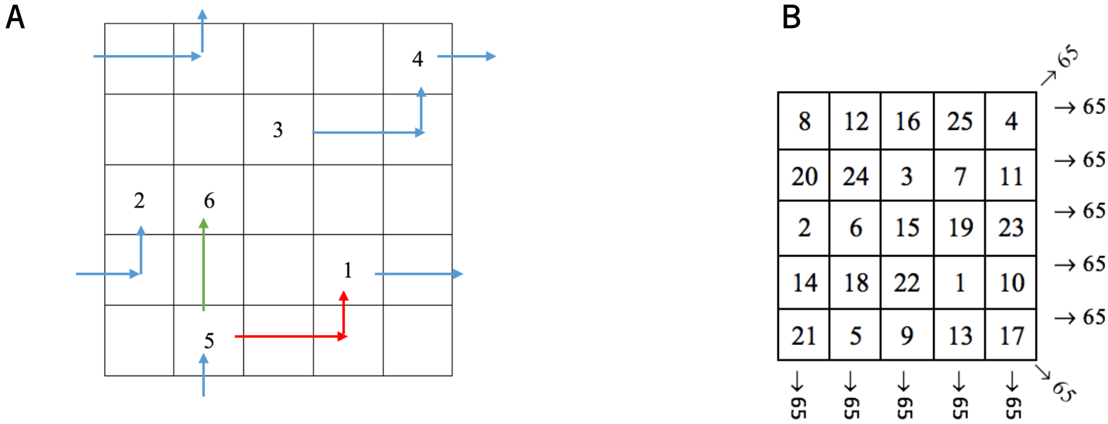

figure 4. A: This is a 5th order square ($n = 5$). The number 1 can be filled in any field. The blue arrows show where to fill in the next number ($a = 2$ fields to the right and $b = 1$ field up). You repeat this step until you come across a field that is already filled (red arrows). In this case have to go $s = 0$ fields to the right, $t = 2$ fields to the right (green arrow), which leads you to a free field. Then you restart with the blue arrows. This is done until the whole square is filled in.
B: A magic square, filled in with the knight's move method.

A filling method from ancient India, which had been known for thousands of years, can also be explained by my results.

The Indian rule is: In the field below the middle field is the number 1. The next number is placed in the adjoining field diagonally to the bottom right. If the next field is already filled, write the next number in the field two below the last filled field.

If you solve the system of equations, which is created by the equations that replace the conditions 6 to 9, you can see that the violet field (figure 5) is the only field that is a solution for the first and second alternative condition. This is exactly the initial field, that is described in the Indian rule.

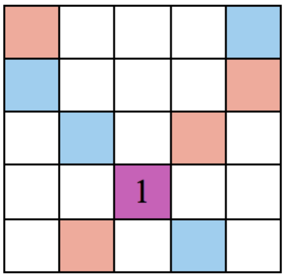

figure 5. The orange marked fields are solutions for the first alternative condition, the blue marked fields are solutions for the second alternative condition. The violet field is the only solution for both conditions. It is the same field, that is described in the Indian rule.

4 Discussion

In my work, I have not only discussed specific filling methods, but have also tried to understand and describe magic squares in a more general context. For me it was very surprising that my mathematical generalization of these objects led to a relatively simple solution. The conditions are easy to check and so you can create many new, yet unknown methods for filling in the numbers yourself. I was also pleased to see that the Indian rule and the knight's move method follow the rules I discovered.

Unfortunately, my results are also limiting. For example, you cannot use them to describe magic squares of an even order. That would certainly be very exciting to investigate in a next step as well as the number of possibilities of magic squares or the behaviour of non-linear filling methods, which means that a, b, s and t do not remain constant during the filling process.

5 Acknowledgements

My special thanks go to Dr. sc. math. Balz Bürgisser, my supervisor, as he was very supportive not only in choosing the topic "magic squares", but also during the whole working process. He always took time for my questions and was himself so fascinated by the topic.

References

(1) Magic Squares of Order Four Philosophical Transactions of the Royal Society of London, Vol. 306 (October 15, 1982), 443-532, Agnes Scott College, https://www.agnesscott.edu/lriddle/women/abstracts/ollerenshaw_Durer.htm (Accessed 2020-10-18)
(2) Karpenko Vladimír. Between Magic and Science: Numerical Magic Squares, 1993, Ambix, 40:3, 121-128, DOI: 10.1179/amb.1993.40.3.121
(3) Sensaio Jaques. Magic Squares- Their History and Construction from Ancient Times to AD 1600, 2019. DOI: 10.1007/978-3-030-17993-9
(4) A Mini-History of Magic Squares, Grogono Magic Squares, https://www.grogono.com/magic/history.php (Accessed 2020-03-15)
(5) Sesiano Jaques. Construction of Magic Squares Using the Knight's Move in Islamic Mathematics. Arch. Hist. Exact Sci., 2003. DOI: 10.1007/s00407-003-0071-4
(6) Karpfinger Christian, Meyberg Kurt. Algebra – Gruppen / Ringe / Körper, 2017, Springer, 4th edition, DOI: 10.1007/978-3-662-54722-9